

EQUIVALENCE OF QUANTUM METRICS WITH A COMMON DOMAIN

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ABSTRACT. We characterize Lipschitz morphisms between quantum compact metric spaces as those $*$ -morphisms which preserve the domain of certain noncommutative analogues of Lipschitz seminorms, namely lower semi-continuous Lip-norms. As a corollary, lower semi-continuous Lip-norms with a shared domain are in fact equivalent. We then note that when a family of lower semi-continuous Lip-norms are uniformly equivalent, then they give rise to totally bounded classes of quantum compact metric spaces, and we apply this observation to several examples of perturbations of quantum metric spaces. We also construct the non-commutative generalization of the Lipschitz distance between quantum compact metric spaces.

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1. INTRODUCTION

Quantum metrics on C^* -algebras, formally provided by generalized Lipschitz seminorms called *Lip-norms* [23, 24], are the seeds for a new analytic framework which brings techniques from metric geometry in C^* -algebra theory and provides a new tool set for problems from mathematical physics, such as finite dimensional approximations of quantum space-time [28, 25, 11, 14, 26, 27, 21], or perturbations of quantum metrics [18, 16]. Quantum compact metric spaces form a natural category [18], whose morphisms are Lipschitz in an appropriate sense. In this paper, we prove that any $*$ -morphism between two quantum compact metric spaces is actually Lipschitz if and only if it is compatible with the domains of the Lip-norms. In particular, $*$ -automorphisms which preserve the domain of a Lip-norm must be

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bi-Lipschitz, and thus all Lip-norms with a common domain are actually equivalent. We then explore three related problems: we show that the topology of pointwise convergence on the group of Lipschitz automorphisms of a quantum compact metric space may be metrized using Lip-norms, and that our previous work on quantum perturbations naturally provide new examples of compact classes of quantum compact metric spaces for the quantum propinquity. We also construct the noncommutative generalization of the Lipschitz distance.

A compact quantum metric space is a generalization of a Lipschitz algebra, inspired by the work of Connes [3, 4] and formalized by Rieffel [23, 24]:

Notation 1.1. The space of self-adjoint elements in a C^* -algebra \mathfrak{A} is denoted by $\mathfrak{sa}(\mathfrak{A})$, while the state space of \mathfrak{A} is denoted by $\mathcal{S}(\mathfrak{A})$. The unit of a unital C^* -algebra \mathfrak{A} is denoted by $1_{\mathfrak{A}}$. Last, we denote the norm on a normed vector space E by $\|\cdot\|_E$ by default.

Last, the diameter of a metric space (E, d) is denoted by $\text{diam}(E, d)$.

Definition 1.2 ([23, 24]). A pair (\mathfrak{A}, L) is a *quantum compact metric space* when \mathfrak{A} is a unital C^* -algebra and L is a seminorm defined on a dense subspace $\text{dom}(L)$ of $\mathfrak{sa}(\mathfrak{A})$, such that:

- (1) $\{a \in \text{dom}(L) : L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
- (2) the *Monge-Kantorovich metric* mk_L defined for any two $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$ by:

$$\text{mk}_L(\varphi, \psi) = \sup \{|\varphi(a) - \psi(a)| : a \in \text{dom}(L), L(a) \leq 1\},$$

metrizes the weak* topology on $\mathcal{S}(\mathfrak{A})$.

If (\mathfrak{A}, L) is a quantum compact metric space, then L is called a *Lip-norm* on \mathfrak{A} .

The classical picture behind Definition (1.2) is provided by a pair $(C(X), \text{Lip})$ of the C^* -algebra of a compact metric space (X, d) and the Lipschitz seminorm Lip associated to the distance function d . A generalization of quantum compact metric spaces to the quantum locally compact setting was proposed in [12, 13].

We may define a category whose objects are quantum compact metric spaces, and whose morphisms are a special type of $*$ -morphisms between the underlying C^* -algebras. There are at least two natural ideas. We may require that a Lipschitz morphism be a $*$ -morphism which is also continuous with respect to the Lip-norms. Formally, if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two quantum compact metric spaces, and $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a $*$ -morphism, this first approach to Lipschitz morphism consists in requiring that there exists $C > 0$ such that $L_{\mathfrak{B}} \circ \varphi \leq CL_{\mathfrak{A}}$. This relation imposes that φ must be unital or null. Indeed, $L_{\mathfrak{B}} \circ \varphi(1_{\mathfrak{A}}) \leq CL_{\mathfrak{A}}(1_{\mathfrak{A}}) = 0$ so $\varphi(1_{\mathfrak{A}}) \in \mathbb{R}1_{\mathfrak{B}}$; since φ is a $*$ -morphism, this leaves us with $\varphi(1_{\mathfrak{A}}) \in \{0, 1_{\mathfrak{B}}\}$. In this paper, we will work with unital $*$ -morphisms.

Alternatively, we may require that the dual map associated to a unital $*$ -morphism be a Lipschitz map between the state spaces equipped with their respective Monge-Kantorovich metric. Continuing with our notations, we would thus ask that there exists $C > 0$ such that for all $\mu, \nu \in \mathcal{S}(\mathfrak{B})$, we have:

$$\text{mk}_{L_{\mathfrak{A}}}(\mu \circ \varphi, \nu \circ \varphi) \leq C \text{mk}_{L_{\mathfrak{B}}}(\mu, \nu).$$

In general, these two notions of a Lipschitz morphisms are not equivalent, owing to the fact that the Monge-Kantorovich metric does not allow the recovery of the

Lip-norm from which it was defined. After all, many Lip-norms may give the same Monge-Kantorovich metric.

However, among all Lip-norms which provide a given Monge-Kantorovich metric, there is a particular one: the largest among all of them, which is characterized as being lower semi-continuous with respect to the norm of the underlying C^* -algebra. In [24], the study of this problem led to the notion of a closed Lip-norm, though the context there was more general (the underlying space was not a C^* -algebra but a more general object called an order-unit space, which may not be complete, leading to some important subtleties).

For our purpose, it is thus natural to work with lower semi-continuous Lip-norms. In this context, our two notions of Lipschitz morphisms coincide. So we summarize our notion by:

Definition 1.3 ([18]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quantum compact metric spaces, with $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$ lower semi-continuous with respect to the norms of, respectively, \mathfrak{A} and \mathfrak{B} . A unital $*$ -morphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is k -Lipschitz for some $k \geq 0$ when $L_{\mathfrak{B}} \circ \varphi \leq k L_{\mathfrak{A}}$, or equivalently:

$$\mu \in \mathcal{S}(\mathfrak{B}) \mapsto \mu \circ \varphi \in \mathcal{S}(\mathfrak{A})$$

is a k -Lipschitz map from $(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})$ to $(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$.

It is easy to check that indeed, composition of Lipschitz morphisms is again Lipschitz, and the identity morphism is 1-Lipschitz, so we have indeed defined a category. It is also easy to check that a k -Lipschitz morphism between two classical compact metric spaces is indeed of the form $f \in C(Y) \mapsto f \circ \theta$ with $\theta : X \rightarrow Y$ is k -Lipschitz.

In this paper, we investigate a third approach of Lipschitz morphisms. If $\text{dom}(L_{\mathfrak{A}})$ and $\text{dom}(L_{\mathfrak{B}})$ are the domains of $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, then a $*$ -morphism may satisfy $\varphi(\text{dom}(L_{\mathfrak{A}})) \subseteq \text{dom}(L_{\mathfrak{B}})$. This appear to be a weaker notion, but as we shall see, it is again equivalent to the notion of a Lipschitz morphism. This reinforces that our notion of a category of quantum compact metric spaces is indeed appropriate.

Our paper then continues with an observation regarding sets of uniformly equivalent Lip-norms. We note that such sets are naturally compact for the quantum propinquity. The quantum Gromov-Hausdorff propinquity [20] is a special member of the family of Gromov-Hausdorff propinquities [17, 15], which are all non-commutative analogues of the Gromov-Hausdorff distance [9, 8] extending the topology of the latter to quantum compact metric spaces. The Gromov-Hausdorff propinquity provides a framework for the geometric study of classes of quantum compact metric spaces. We have established, for instance, the continuity of various natural families of C^* -algebras such as quantum tori [14] or certain AF-algebras [1]. We constructed finite dimensional approximations for these spaces as well, answering some informal statements in mathematical physics. Another recent advance was the generalization of Gromov's theorem to our propinquity [19], providing us with an insight into the topological properties of various sets of compact quantum metric spaces.

In general, proving that a class of quantum compact metric spaces is indeed totally bounded for the propinquity may be subtle. For instance, [19] relies on finite dimensional approximations, which are themselves a challenge. Nonetheless, as

seen in [1, Theorem 6.3], our generalized Gromov's theorem can be put to use. Another approach taken in [1] was the construction of continuous maps from some compact spaces onto classes of quantum compact metric spaces; this method also applies to quantum tori and their finite dimensional approximations [11, 14] and we shall see that it applies to conformal deformations as well in this paper. In this paper, we take yet another road to establish the compactness of some interesting classes of quantum compact metric spaces, obtained as perturbations of given quantum metrics.

Prior to the introduction of noncommutative Gromov-Hausdorff distances [28, 10, 20, 17, 15], the idea of perturbations for a quantum metrics seemed to largely rely on the informal idea that certain algebraic expressions are qualitatively close to some original metric. We recently formalized this idea by actually establishing bounds on how far, in the sense of the propinquity, certain particular perturbations actually are. Examples of such perturbations include conformal deformations [22] of quantum metrics arising from certain spectral triples [18], leading to twisted spectral triples introduced by Connes and Moscovici [5]. We also brought curved quantum tori of Sitarz and Dąbrowski [6, 7] into our program [16]. We shall see that a core ingredient of the constructions of these perturbations provide a uniform equivalence between Lip-norms, which in turn gives a compactness result.

We note that, besides the Gromov-Hausdorff distance, a standard extended metric between compact metric spaces is the Lipschitz distance. We provide in this paper a construction for the noncommutative version of the Lipschitz distance, which fits very well with the picture of Lipschitz morphisms presented in this paper.

The last section of this paper presents a natural metric on Lipschitz morphisms, built from quantum metrics.

2. EQUIVALENCE OF LIP-NORMS AND LIPSCHITZ MORPHISMS

Quantum metrics defined on the same domain are, in fact, equivalent, under a natural technical condition, which may always be assumed to no cost to the underlying metric structure. This observation is the subject of the following theorem, and constitutes our main result.

Theorem 2.1. *Let (\mathfrak{A}, L) be a quantum compact metric space, with L lower semi-continuous with domain $\text{dom}(L)$. Let S be a seminorm on $\text{dom}(L)$ such that:*

- (1) *S is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$,*
- (2) *$S(1_{\mathfrak{A}}) = 0$.*

Then there exists $C > 0$ such that for all $a \in \text{dom}(L)$:

$$S(a) \leq CL(a).$$

Proof. Our proof proceeds in three steps. First, we prove that the domain of a lower semi-continuous seminorm can be made naturally into a Banach space. Then, we use the open mapping theorem to show that different lower semi-continuous seminorms defined on the same domain give rise to equivalent Banach norms with the construction in our first step. Last, we conclude our theorem.

Step 1. Let S be a lower semi-continuous seminorm defined on some dense subspace $\text{dom}(S)$ of $\mathfrak{sa}(\mathfrak{A})$. Let $\|\cdot\|_S = \|\cdot\|_{\mathfrak{A}} + S$. We first check that $\text{dom}(S)$ is a Banach space for the norm $\|\cdot\|_S$.

It is straightforward that $\|\cdot\|_S$ is a norm on $\text{dom}(S)$.

Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\text{dom}(S)$ for $\|\cdot\|_S$. Thus $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for $\|\cdot\|_{\mathfrak{A}}$, which is complete, so $(a_n)_{n \in \mathbb{N}}$ converges to some $a \in \mathfrak{sa}(\mathfrak{A})$ for $\|\cdot\|_{\mathfrak{A}}$.

We also observe that $(a_n)_{n \in \mathbb{N}}$ is Cauchy, hence bounded, for S ; thus there exists $M > 0$ such that $S(a_n) \leq M$ for all $n \in \mathbb{N}$. As S is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$, we obtain that $S(a) \leq \liminf_{n \rightarrow \infty} S(a_n) \leq M$. Thus $a \in \text{dom}(S)$.

Let $\varepsilon > 0$. Since $(a_n)_{n \in \mathbb{N}}$ is Cauchy for S , there exists $N \in \mathbb{N}$ such that for all $p, q \geq N$ we have $S(a_p - a_q) \leq \varepsilon$. Since S is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$, we thus have, for all $p \geq N$:

$$S(a - a_p) \leq \liminf_{q \rightarrow \infty} S(a_p - a_q) \leq \varepsilon.$$

Thus $\lim_{p \rightarrow \infty} S(a - a_p) = 0$. This proves that $\lim_{n \rightarrow \infty} \|a - a_n\|_S = 0$ as desired.

Step 2. If L and S be two lower semi-continuous seminorms on some common dense subspace $\text{dom}(L)$ of $\mathfrak{sa}(\mathfrak{A})$, then the norms:

$$\|\cdot\|_L = \|\cdot\|_{\mathfrak{A}} + L \text{ and } \|\cdot\|_S = \|\cdot\|_{\mathfrak{A}} + S$$

are equivalent.

The norms $\|\cdot\|_L$ and $\|\cdot\|_S$ both make $\text{dom}(L)$ into a Banach space, by step 1.

We begin with a simple observation. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\text{dom}(L)$ which converges for both $\|\cdot\|_L$ and $\|\cdot\|_S$. Let $x \in \text{dom}(L)$ be the limit of $(x_n)_{n \in \mathbb{N}}$ for $\|\cdot\|_L$ and $y \in \text{dom}(L)$ be the limit of $(x_n)_{n \in \mathbb{N}}$ for $\|\cdot\|_S$. We note that in particular, $(x_n)_{n \in \mathbb{N}}$ converges to both x and y for $\|\cdot\|_{\mathfrak{A}}$. Thus $x = y$.

Let now $\|\cdot\|_* = \|\cdot\|_L + \|\cdot\|_S$. If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for $\|\cdot\|_*$, then the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy for both $\|\cdot\|_L$ and $\|\cdot\|_S$ and thus converges for both these norms, since they are complete; by our previous observation, $(x_n)_{n \in \mathbb{N}}$ has the same limit $x \in \text{dom}(L)$ for both these norms. Hence, $(x_n)_{n \in \mathbb{N}}$ converges to x for $\|\cdot\|_*$, i.e. $(\text{dom}(L), \|\cdot\|_*)$ is a Banach space.

Now, since $\|\cdot\|_L \leq \|\cdot\|_*$, the open mapping theorem [2] implies that there exists $k > 0$ such that $\|\cdot\|_* \leq k\|\cdot\|_L$. We then conclude easily that $\|\cdot\|_S \leq k\|\cdot\|_L$. Similarly, for some $k' > 0$, we have $\|\cdot\|_L \leq k'\|\cdot\|_S$.

Step 3. We now conclude our theorem.

We thus are given a lower semi-continuous Lip-norm L on \mathfrak{A} , and some seminorm S on the domain of L , with $S(1_{\mathfrak{A}}) = 0$ and S lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

Using our previous step, there exists $k > 0$ such that for all $a \in \text{dom}(L)$, we have:

$$\begin{aligned} S(a) &= \|a\|_S - \|a\|_{\mathfrak{A}} \\ &\leq k\|a\|_L - \|a\|_{\mathfrak{A}} \\ &\leq k(\|a\|_{\mathfrak{A}} + L(a)) - \|a\|_{\mathfrak{A}} \\ &= (k-1)\|a\|_{\mathfrak{A}} + kL(a). \end{aligned}$$

Let $D = \text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_L)$ (which is finite since L is a Lip-norm). Let $a \in \text{dom}(L_{\mathfrak{A}})$. As L is a Lip-norm, there exists $t \in \mathbb{R}$ such that:

$$(2.1) \quad \|a + t1_{\mathfrak{A}}\|_{\mathfrak{A}} \leq DL(a).$$

Indeed, let $\mu \in \mathcal{S}(\mathfrak{A})$. For all $\nu \in \mathcal{S}(\mathfrak{A})$ then:

$$|\mu(a) - \nu(a)| \leq L(a) \text{mk}_L(\mu, \nu) \leq L(a)D.$$

Of course, $L(a + t1_{\mathfrak{A}}) = L(a)$ and $S(a + t1_{\mathfrak{A}}) = S(a)$ since $S(1_{\mathfrak{A}}) = 0$. Thus:

$$\begin{aligned} S(a) &= S(a + t1_{\mathfrak{A}}) \\ &\leq (k-1)\|a + t1_{\mathfrak{A}}\|_{\mathfrak{A}} + kL(a + t1_{\mathfrak{A}}) \\ &\leq (k-1)DL(a) + kL(a) = ((k-1)D + k)L(a). \end{aligned}$$

This concludes our proof, with $C = ((k-1)D + k)$. \square

Theorem (2.1) has the following consequences:

Corollary 2.2. *Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two compact quantum metric spaces whose Lip-norms are lower semi-continuous. Let $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ be some unital *-morphism such that $\varphi(\text{dom}(L_{\mathfrak{A}})) \subseteq \text{dom}(L_{\mathfrak{B}})$. Then there exists $C > 0$ such that:*

$$L_{\mathfrak{B}} \circ \varphi \leq CL_{\mathfrak{A}}$$

i.e. φ is a C -Lipschitz morphism.

Proof. Let $S = L_{\mathfrak{B}} \circ \varphi$. We note that S is a lower semi-continuous seminorm which takes finite values on $\text{dom}(L_{\mathfrak{A}})$ by assumption. Moreover $S(1_{\mathfrak{A}}) = 0$. Thus our corollary follows from Theorem (2.1). \square

Corollary 2.3. *Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quantum compact metric spaces whose Lip-norms are lower semi-continuous. If $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a *-isomorphism such that $\varphi(\text{dom}(L_{\mathfrak{A}})) = \text{dom}(L_{\mathfrak{B}})$, then there exists $C > 0$ such that:*

$$C^{-1}L_{\mathfrak{A}} \leq L_{\mathfrak{B}} \circ \varphi \leq CL_{\mathfrak{A}}.$$

Proof. This follows from Corollary (2.2). \square

Corollary 2.4. *Let (\mathfrak{A}, L) be a quantum compact metric space where L is lower semi-continuous. Let α be a *-automorphism of \mathfrak{A} . The following two assertions are equivalent:*

- (1) $\alpha(\text{dom}(L)) = \text{dom}(L)$,
- (2) there exists $C > 0$ such that $C^{-1}L \circ \alpha \leq L \leq CL \circ \alpha$.

Proof. Assume (2) first. Let $a \in \text{dom}(L)$. Then:

$$L(\alpha^{-1}(a)) \leq CL(\alpha(\alpha^{-1}(a))) = L(a) < \infty,$$

hence $\alpha^{-1}(a) \in \text{dom}(L)$. Thus $a = \alpha(\alpha^{-1}(a)) \in \alpha(\text{dom}(L))$, i.e. $\text{dom}(L) \subseteq \alpha(\text{dom}(L))$. The converse inclusion is proven similarly.

Assume (1). Then (2) follows from Corollary (2.3). \square

Corollary 2.5. *Let \mathfrak{A} be a unital C^* -algebra and L_1, L_2 be two lower semi-continuous Lip-norms on \mathfrak{A} . The following assertions are equivalent:*

- (1) L_1 and L_2 are equivalent,
- (2) $\text{dom}(L_1) = \text{dom}(L_2)$.

Proof. Apply Corollary (2.4) to the identity automorphism of \mathfrak{A} . \square

3. COMPACTNESS OF CLASSES OF PERTURBATIONS OF QUASI-LEIBNIZ QUANTUM COMPACT METRIC SPACES

To present the result in this section, we introduce a simple metric on Lip-norms over a fixed C^* -algebra.

Definition 3.1. For any two lower semi-continuous Lip-norms L, L' on a unital C^* -algebra \mathfrak{A} , we define $\text{Haus}_{\mathfrak{A}}^{\circ}(L, L')$ by:

$$\text{Haus}_{\mathfrak{A}}^{\circ}(L, L') = \text{Haus}_{\|\cdot\|_{\mathfrak{A}}}(\{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) \leq 1\}, \{a \in \mathfrak{sa}(\mathfrak{A}) : L'(a) \leq 1\}),$$

where $\text{Haus}_{\|\cdot\|_{\mathfrak{A}}}$ is the Hausdorff distance induced on closed subsets of \mathfrak{A} by the norm $\|\cdot\|_{\mathfrak{A}}$.

We begin by observing that our distance $\text{Haus}_{\mathfrak{A}}^{\circ}$ is indeed finite. Let us start by recalling the following fundamental characterization of quantum compact metric spaces proved by Rieffel in [23], and akin to a noncommutative Arzela-Ascoli theorem:

Theorem 3.2 ([23, 24]). *Let \mathfrak{A} be a unital C^* -algebra and L a seminorm defined on a dense subspace $\text{dom}(L)$ of $\mathfrak{sa}(\mathfrak{A})$ and such that $\{a \in \text{dom}(L) : L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$. The following assertions are equivalent:*

- (1) (\mathfrak{A}, L) is a quantum compact metric space,
- (2) $\{a \in \text{dom}(L) : L(a) \leq 1, \varphi(a) = 0\}$ is totally bounded in norm for some state $\varphi \in \mathcal{S}(\mathfrak{A})$,
- (3) $\{a \in \text{dom}(L) : L(a) \leq 1, \varphi(a) = 0\}$ is totally bounded in norm for all states $\varphi \in \mathcal{S}(\mathfrak{A})$,
- (4) $\{a \in \text{dom}(L) : L(a) \leq 1, \|a\|_{\mathfrak{A}} \leq 1\}$ is totally bounded in norm.

In particular, if L is lower semi-continuous, then L is a Lip-norm if, and only if any of the sets above are compact.

Lemma 3.3. *Let \mathfrak{A} be a unital C^* -algebra. For any two lower semi-continuous Lip-norms L_1 and L_2 on \mathfrak{A} , and for any state $\varphi \in \mathcal{S}(\mathfrak{A})$, we have:*

$$\begin{aligned} \text{Haus}_{\mathfrak{A}}^{\circ}(L_1, L_2) \leq \text{Haus}_{\|\cdot\|_{\mathfrak{A}}}(\{a \in \mathfrak{sa}(\mathfrak{A}) : \varphi(a) = 0, L_1(a) \leq 1\}, \\ \{a \in \mathfrak{sa}(\mathfrak{A}) : \varphi(a) = 0, L_2(a) \leq 1\}). \end{aligned}$$

In particular, $\text{Haus}_{\mathfrak{A}}^{\circ}(L_1, L_2)$ is finite.

Proof. Let $\varphi \in \mathcal{S}(\mathfrak{A})$. By Theorem (3.2), $\{a \in \mathfrak{sa}(\mathfrak{A}) : \varphi(a) = 0, L_j(a) \leq 1\}$ are compact for $j = 1, 2$. Thus the Hausdorff distance (for the norm of \mathfrak{A}) between these two sets is finite; let us denote it by d .

Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $L_1(a) \leq 1$. There exists $b \in \mathfrak{sa}(\mathfrak{A})$ with $L_2(b) \leq 1$ and $\varphi(b) = 0$ such that $\|a - \varphi(a)1_{\mathfrak{A}} - b\|_{\mathfrak{A}} \leq d$. Thus $\|a - (b + \varphi(a)1_{\mathfrak{A}})\|_{\mathfrak{A}} \leq d$, and we note that $L_2(b + \varphi(a)1_{\mathfrak{A}}) = L_2(b) \leq 1$. As the argument is symmetric in L_1 and L_2 , we have shown our lemma. \square

Our goal is to establish a new sufficient condition for certain classes of quantum compact metric spaces to be totally bounded for the *quantum propinquity*. We refer to [20, 17, 15, 19, 18] for the definition of the quantum propinquity and some of its properties. We briefly recall from [20] the notion of a bridge, as it will be used in our next proof, and provide a characterization of the quantum propinquity.

Let \mathfrak{A} and \mathfrak{B} be two unital C^* -algebras. A *bridge* $(\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from \mathfrak{A} to \mathfrak{B} is a unital C^* -algebra \mathfrak{D} and two unital $*$ -monomorphisms $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$, as well as an element $\omega \in \mathfrak{D}$ such that for at least one state φ of \mathfrak{D} , we have $\varphi(\omega d) = \varphi(d\omega) = \varphi(d)$ for all $d \in \mathfrak{D}$. The set of all such states of \mathfrak{D} , denoted by $\mathcal{S}_1(\mathfrak{D}|\omega)$, is the 1-level set of \mathfrak{D} .

Now if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two quantum compact metric spaces, then we can associate a number, called the *length*, to a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from \mathfrak{A} to \mathfrak{B} . We first define the *reach* of γ as the Hausdorff distance between $\{\pi_{\mathfrak{A}}(a)\omega : a \in \mathfrak{sa}(\mathfrak{A}), L_{\mathfrak{A}}(a) \leq 1\}$ and $\{\omega\pi_{\mathfrak{B}}(b) : b \in \mathfrak{sa}(\mathfrak{B}), L_{\mathfrak{B}}(b) \leq 1\}$ for $\text{Haus}_{\|\cdot\|_{\mathfrak{D}}}$. We then define the *height* of γ as the maximum of the Hausdorff distance, for $\text{Haus}_{\text{mk}_{L_{\mathfrak{A}}}}$, between $\mathcal{S}(\mathfrak{A})$ and $\{\varphi \circ \pi_{\mathfrak{A}} : \varphi \in \mathcal{S}_1(\mathfrak{D}|\omega)\}$, and the Hausdorff distance for $\text{Haus}_{\text{mk}_{L_{\mathfrak{B}}}}$, between $\mathcal{S}(\mathfrak{B})$ and $\{\varphi \circ \pi_{\mathfrak{B}} : \varphi \in \mathcal{S}_1(\mathfrak{D}|\omega)\}$.

The length $\lambda(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$ of the bridge γ is the maximum of its reach and its height. The quantum propinquity is constructed from bridges, although it requires a few technical steps. In particular, the quantum propinquity is defined on classes of F -quasi-Leibniz quantum compact metric space for an *admissible function* F , i.e. a function $F : [0, \infty)^4 \rightarrow [0, \infty)$ which is increasing for the product order on $[0, \infty)^4$ and such that $F(x, y, l_x, l_y) \geq xl_y + yl_x$ for all $x, y, l_x, l_y \geq 0$. Given such a function, a F -quasi-Leibniz quantum compact metric space (\mathfrak{A}, L) is a quantum compact metric space such that for all $a, b \in \mathfrak{sa}(\mathfrak{A})$ we have:

$$\max \left\{ L \left(\frac{ab + ba}{2} \right), L \left(\frac{ab - ba}{2i} \right) \right\} \leq F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, L(a), L(b)).$$

The following result characterizes the quantum propinquity.

Theorem-Definition 3.4 ([20]). *Let \mathcal{L} be the class of all F -quasi-Leibniz quantum compact metric spaces for some admissible function F . There exists a class function Λ_F from $\mathcal{L} \times \mathcal{L}$ to $[0, \infty) \subseteq \mathbb{R}$ such that:*

- (1) *for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ we have:*

$$0 \leq \Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \max \{ \text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}}), \text{diam}(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}}) \},$$
- (2) *for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ we have:*

$$\Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \Lambda_F((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{A}, L_{\mathfrak{A}})),$$
- (3) *for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}}) \in \mathcal{L}$ we have:*

$$\Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{C}, L_{\mathfrak{C}})) \leq \Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) + \Lambda_F((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}})),$$
- (4) *for all $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ and for any bridge γ from \mathfrak{A} to \mathfrak{B} , we have:*

$$\Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \lambda(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}}),$$
- (5) *for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$, we have $\Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = 0$ if and only if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are isometrically isomorphic, i.e. if and only if there exists a $*$ -isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ with $L_{\mathfrak{B}} \circ \pi = L_{\mathfrak{A}}$, or equivalently there exists a $*$ -isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ whose dual map π^* is an isometry from $(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})$ into $(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$,*
- (6) *if Ξ is a class function from $\mathcal{L} \times \mathcal{L}$ to $[0, \infty)$ which satisfies Properties (2), (3) and (4) above, then $\Xi((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$ for all $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ in \mathcal{L} ,*

- (7) the topology induced by Λ_F on the class of classical metric spaces agrees with the topology induced by the Gromov-Hausdorff distance.

We connect our new distance between Lip-norms on a fixed C^* -algebra and the propinquity easily.

Proposition 3.5. *Let \mathfrak{A} be a unital C^* -algebra. If L_1 and L_2 are two F -quasi-Leibniz lower semi-continuous Lip-norms on \mathfrak{A} for some admissible function F , then:*

$$\Lambda_F((\mathfrak{A}, L_1), (\mathfrak{A}, L_2)) \leq \text{Haus}_{\mathfrak{A}}^{\circ}(L_1, L_2).$$

Proof. We simply use the bridge $(\mathfrak{A}, 1_{\mathfrak{A}}, \text{id}, \text{id})$ where id is the identity on \mathfrak{A} . \square

The purpose of this section is to establish the fact that uniformly equivalent Lip-norms, as defined in the hypothesis of the next proposition, provide totally bounded classes of quantum compact metric spaces for the metric Haus° and thus for the quantum propinquity, whenever applicable.

Proposition 3.6. *Let (\mathfrak{A}, L) be a quantum compact metric space where L is lower semi-continuous. If Ξ is a set of lower semi-continuous Lip-norms on \mathfrak{A} for which there exists $C > 0$ such that, for all $\text{Lip} \in \Xi$, we have $L \leq C \text{Lip}$, then Ξ is totally bounded for $\text{Haus}_{\mathfrak{A}}^{\circ}$ (and therefore, when applicable, for the quantum propinquity as well).*

Proof. We fix $\mu \in \mathcal{S}(\mathfrak{A})$. By assumption, for all $\text{Lip} \in \Xi$, we have:

$$\{a \in \mathfrak{sa}(\mathfrak{A}) : \text{Lip}(a) \leq 1, \mu(a) = 0\} \subseteq \{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) \leq C, \mu(a) = 0\}.$$

Now, since L is a lower semi-continuous Lip-norm, the set:

$$\mathfrak{L} = \{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) \leq C, \mu(a) = 0\}$$

is compact for $\|\cdot\|_{\mathfrak{A}}$ by Theorem (3.2). Thus by Blaschke's Theorem, the hyperspace of the closed subsets of \mathfrak{L} is compact for the Hausdorff distance $\text{Haus}_{\|\cdot\|_{\mathfrak{A}}}$. We conclude our proof using Lemma (3.3) (and Proposition (3.5) for the quantum propinquity conclusion). \square

The main application of Proposition (3.6) in this paper concerns certain perturbations we have established in [19, 18]. These perturbations were constructed using [18, Lemma 3.79], which encapsulates a recurrent computation when estimating the quantum propinquity between two quasi-Leibniz quantum compact metric spaces. When using this lemma, one will typically obtain uniformly equivalent families of Lip-norms, thus the following examples will be typical.

Notation 3.7. If \mathcal{H} is some Hilbert space and T is a bounded linear operator on \mathcal{H} , then we denote the norm of T by $\|T\|_{\mathcal{H}}$.

Example 3.8 (Bounded perturbations).

Proposition 3.9 ([18, Proposition 3.82]). *Let \mathfrak{A} be a unital C^* -algebra, π a unital faithful $*$ -representation of \mathfrak{A} on some Hilbert space \mathcal{H} , and D a self-adjoint, possibly unbounded operator on \mathcal{H} such that setting:*

$$L : a \in \mathfrak{A} \mapsto \| [D, \pi(a)] \|_{\mathcal{H}}$$

and allowing for L to assume the value ∞ , the pair (\mathfrak{A}, L) is a Leibniz quantum compact metric space.

Let \mathfrak{B} be the C^* -algebra of all bounded linear operators on \mathcal{H} .
For any $\omega \in \mathfrak{sa}(\mathfrak{B})$, we define:

$$D_\omega = D + \omega \text{ and } L_\omega : a \in \mathfrak{A} \mapsto ||| [D_\omega, \pi(a)] |||_{\mathcal{H}}.$$

The pair (\mathfrak{A}, L_ω) is a Leibniz quantum compact metric space for all bounded self-adjoint ω on \mathcal{H} and, moreover:

$$\omega \in \mathfrak{sa}(\mathfrak{B}) \longmapsto (\mathfrak{A}, L_\omega)$$

is continuous for the quantum Gromov-Hausdorff propinquity Λ .

We shall henceforth refer to the proof of [18, Proposition 3.82]. We observe that key to that argument is the observation that for all $a \in \mathfrak{sa}(\mathfrak{A})$ we have:

$$L(a) \leq (1 + r\|\omega\|_{\mathfrak{B}})\|\omega - 1_{\mathfrak{B}}\|_{\mathfrak{B}}L_\omega(a)$$

and thus we see that if $\Omega = \{\omega \in \mathfrak{sa}(\mathfrak{B}) : \|\omega\|_{\mathfrak{B}} \leq C\}$ for some $C > 0$, then the set of quantum metrics:

$$\{L_\omega : \omega \in \Omega\}$$

is totally bounded by Proposition (3.6) for both $\text{Haus}_{\mathfrak{A}}^\circ$ and Λ . Yet Ω is not totally bounded unless \mathfrak{B} is finite dimensional.

A deeper example of a non trivial class of totally bounded quantum metrics obtained from perturbations and [19, Lemma 3.79] is given by curved quantum tori, where, once more, the space of parameters is not totally bounded. These examples come from mathematical physics and were treated in [16] from the metric perspective.

Example 3.10 (Curved quantum tori).

Theorem 3.11 ([16, Theorem III.1]). Let \mathfrak{A} be a unital C^* -algebra, and let α be a strongly continuous ergodic action of a compact Lie group G on \mathfrak{A} . Let n be the dimension of G . We endow the dual \mathfrak{g}' of the Lie algebra \mathfrak{g} of G with an inner product $\langle \cdot, \cdot \rangle$, and we denote by C the Clifford algebra of $(\mathfrak{g}', \langle \cdot, \cdot \rangle)$. Let c be a faithful nondegenerate representation of C on some Hilbert space \mathcal{H}_C .

We fix some orthonormal basis $\{e_1, \dots, e_n\}$ of \mathfrak{g}' , and we let $X_1, \dots, X_n \in \mathfrak{g}$ be the dual basis. For each $j \in \{1, \dots, n\}$, we define the derivation δ_j of \mathfrak{A} via α and X_j , by:

$$\partial_j : a \longmapsto \lim_{h \rightarrow 0} \frac{\alpha_{\exp(hX_j)}(a) - a}{h}$$

wherever defined. Let \mathfrak{A}^1 be the common domain of $\partial_1, \dots, \partial_n$, which is a dense $*$ -subalgebra in \mathfrak{A} .

Let τ be the unique α -invariant tracial state of \mathfrak{A} . Let ρ be the representation of \mathfrak{A} obtained from the Gel'fand-Naimark-Segal construction applied to τ and let $L^2(\mathfrak{A}, \tau)$ be the corresponding Hilbert space. As \mathfrak{A}^1 is dense in $L^2(\mathfrak{A}, \tau)$, the operator ∂_j defines an unbounded densely defined operator on $L^2(\mathfrak{A}, \tau)$ for all $j \in \{1, \dots, n\}$.

Let $\mathcal{H} = L^2(\mathfrak{A}, \tau) \overline{\otimes} \mathcal{H}_C$ where $\overline{\otimes}$ is the standard tensor product for Hilbert spaces. We define the following representation of \mathfrak{A} on \mathcal{H} :

$$\pi(a) : b \otimes f \longmapsto \rho(a)b \otimes f.$$

Let:

$$H = \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix}$$

where for all $j, k \in \{1, \dots, n\}$, the coefficients h_{jk} are elements in the commutant of \mathfrak{A} in $L^2(\mathfrak{A}, \tau)$, and where H is invertible as an operator on $\mathcal{H}' = \oplus_{j=1}^n L^2(\mathfrak{A}, \tau)$. We denote the identity over \mathcal{H}' by $1_{\mathcal{H}'}$.

We define:

$$D_H = \sum_{j=1}^n \sum_{k=1}^n h_{kj} \partial_k \otimes c(e_j)$$

so that for all $a \in \mathfrak{A}$:

$$[D_H, \pi(a)] = \sum_{j=1}^n \sum_{k=1}^n h_{kj} \rho(\partial_k(a)) \otimes c(e_j).$$

We define:

$$\mathbb{L}_H(a) = ||| [D_H, \pi(a)] |||_{\mathcal{H}'}$$

Then:

- (1) $(\mathfrak{A}, \mathbb{L}_H)$ is a Leibniz quantum compact metric space,
- (2) if we set:

$$H' = \begin{pmatrix} h'_{11} & \cdots & h'_{1n} \\ \vdots & & \vdots \\ h'_{n1} & \cdots & h'_{nn} \end{pmatrix}$$

where h'_{jk} lies in the commutant of $\rho(\mathfrak{A})$ for all $j, k \in \{1, \dots, n\}$ and where H' is invertible as an operator on \mathcal{H}' , then:

$$\begin{aligned} \Lambda((\mathfrak{A}, \mathbb{L}_H), (\mathfrak{A}, \mathbb{L}_{H'})) &\leq n \max \left\{ ||| 1_{\mathcal{H}'} - H' H^{-1} |||_{\mathcal{H}'}, ||| 1_{\mathcal{H}'} - H H'^{-1} |||_{\mathcal{H}'} \right\} \\ &\times \left[1 + \frac{1}{2} \max \left\{ \left(1 + n ||| 1 - H^{-1} |||_{\mathcal{H}'} \right)^{-1}, \right. \right. \\ &\quad \left. \left. \left(1 + n ||| 1 - H'^{-1} |||_{\mathcal{H}'} \right)^{-1} \right\} \text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_L) \right]. \end{aligned}$$

We thus may apply Proposition (3.6) when we restrict the parameter space $\mathcal{H}_C = \{H : ||| H |||_{\mathfrak{B}}, ||| H^{-1} |||_{\mathfrak{B}} \leq C\}$ for any $C > 0$. Of course this set itself is not totally bounded, yet the space $\{\mathbb{L}_H : H \in \mathcal{H}_C\}$ is totally bounded for both $\text{Haus}_{\mathfrak{A}}^{\mathfrak{A}}$ and Λ .

We conclude this section with another example of a compact class of quantum compact metric spaces obtained from perturbations. We include this example as another compact class of quantum metric spaces, derived from [19, Lemma 3.79], though in this case it does not require Proposition (3.6).

Example 3.12 (Conformal Perturbations). We proved in [18] that small conformal perturbations of quantum metrics are indeed close for the quantum propinquity. We recall the result here to fix our notations.

Theorem 3.13 ([18, Theorem 3.81]). *Let \mathfrak{A} be a unital C^* -algebra, π a faithful unital $*$ -representation of \mathfrak{A} on some Hilbert space \mathcal{H} and D be a not necessarily bounded self-adjoint operator on \mathcal{H} such that if $L : a \in \mathfrak{sa}(\mathfrak{A}) \mapsto ||| [D, \pi(a)] |||_{\mathfrak{B}}$ then (\mathfrak{A}, L) is a Leibniz quantum compact metric space.*

Let $\text{GLip}(\mathfrak{A})$ be the set of all invertible elements h in $\mathfrak{sa}(\mathfrak{A})$ with $L(h) < \infty$. For any $h \in \text{GLip}(\mathfrak{A})$, we define $D_h = \pi(h)D\pi(h)$, $\sigma_h : a \in \mathfrak{A} \mapsto h^2 a h^{-2}$ and:

$$L_h : a \in \mathfrak{sa}(\mathfrak{A}) \mapsto ||| D_h \pi(a) - \pi(\sigma_h(a)) D_h |||_{\mathfrak{B}}.$$

Then (\mathfrak{A}, L_h) is a $(||h^2||_{\mathfrak{A}} ||h^{-2}||_{\mathfrak{A}}, 0)$ -quasi-Leibniz quantum compact metric space and moreover, if $(h_n)_{n \in \mathbb{N}}$ is a sequence in $\text{GLip}(\mathfrak{A})$ which converges to $h \in \text{GLip}(\mathfrak{A})$ and such that:

$$\lim_{n \rightarrow \infty} L(h_n^{-1} h) = \lim_{n \rightarrow \infty} L(h_n h^{-1}) = 0,$$

then:

$$\lim_{n \rightarrow \infty} \Lambda_{F_M}((\mathfrak{A}, L_{h_n}), (\mathfrak{A}, L_h)) = 0,$$

where $M \geq \sup_{n \in \mathbb{N}} ||h_n^2||_{\mathfrak{A}} ||h_n^{-2}||_{\mathfrak{A}}$ and $F_M : x, y, u, v \in [0, \infty) \mapsto M(xv + yu)$.

Let $K_1, K_2, K_3 > 0$ and define:

$$\Omega_{K_1, K_2, K_3} = \left\{ \omega \in \mathfrak{sa}(\mathfrak{A}) : L(\omega) \leq K_1, \|\omega\|_{\mathfrak{A}} \leq K_2, \|\omega^{-1}\|_{\mathfrak{A}} \leq K_3 \right\}.$$

Then Ω_{K_1, K_2, K_3} is compact for $\|\cdot\|_{\mathfrak{A}}$ since L is a lower semi-continuous Lip-norm. Theorem (3.13) shows that conformal perturbations are continuous for the quantum propinquity, and thus, using our notations, $\{L_\omega : \omega \in \Omega_{K_1, K_2, K_3}\}$ is compact.

4. LIPSCHITZ DISTANCE BETWEEN COMPACT QUANTUM METRIC SPACES

The Lipschitz distance between compact metric spaces [9] provides a distance between homeomorphic compact metric spaces based upon bi-Lipschitz isomorphisms, and thus it is natural to define it in this paper in light of our study of Lipschitz morphisms.

This section provides the noncommutative generalization of the Lipschitz metric, which in essence is a metric on Lip-norms with common domains. The quantum Lipschitz distance is complete and dominates the quantum propinquity when working on appropriate classes of quasi-Leibniz quantum compact metric spaces. The Lipschitz distance also provides natural examples of totally bounded classes for the quantum propinquity, and thus compact classes for the dual propinquity [17].

Notation 4.1. Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quantum compact metric spaces and let $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ a unital $*$ -morphism. We denote by $\text{dil}(\varphi)$ the Lipschitz seminorm of the dual map $\varphi : \mu \in (\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}}) \mapsto \mu \circ \varphi \in (\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$, i.e.:

$$\text{dil}(\varphi) = \sup \left\{ \frac{\text{mk}_{L_{\mathfrak{B}}}(\mu \circ \varphi, \nu \circ \varphi)}{\text{mk}_{L_{\mathfrak{A}}}(\mu, \nu)} : \mu, \nu \in \mathcal{S}(\mathfrak{B}), \mu \neq \nu \right\},$$

with the understanding that this quantity may be infinite. We refer to this quantity as the *dilation factor*, or just *dilation* of the given Lipschitz morphism.

Remark 4.2. If $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two quantum compact metric spaces with lower semicontinuous Lip-norms, and if $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ a unital $*$ -morphism, then $\text{dil}(\varphi) = \inf \{C > 0 : L_{\mathfrak{B}} \circ \varphi \leq CL_{\mathfrak{A}}\}$ with the usual convention that $\inf \emptyset = \infty$.

Definition 4.3. The *Lipschitz distance* between two quantum compact metric spaces $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ is:

$$\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \inf \left\{ \max \left\{ |\ln(\text{dil}(\varphi))|, |\ln(\text{dil}(\varphi^{-1}))| \right\} : \varphi : \mathfrak{A} \rightarrow \mathfrak{B} \text{ is a } *\text{-isomorphism} \right\},$$

with the conventions that $\inf \emptyset = \infty$ and $\ln(\infty) = \infty$.

Proposition 4.4. If $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two quantum compact metric spaces with lower semicontinuous Lip-norms. Then:

$$\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \inf \left\{ \max \left\{ |\ln(\text{dil}(\varphi))|, |\ln(\text{dil}(\varphi^{-1}))| \right\} : \begin{array}{l} \varphi : \mathfrak{A} \rightarrow \mathfrak{B} \text{ is a } *\text{-isomorphism} \\ \varphi(\text{dom}(L_{\mathfrak{A}})) = \text{dom}(L_{\mathfrak{B}}) \end{array} \right\},$$

with the convention that $\inf \emptyset = \infty$.

Proof. This follows from Corollary (2.3). \square

A natural class for the study of the Lipschitz distance is given by the following definitions, which includes all quasi-Leibniz quantum compact metric spaces: we simply require lower semi-continuity of the quantum metrics, as it fits the general framework of this paper, and we require that the domain of the Lip-norm is a Jordan-Lie algebra, to retain a minimum amount of information on the multiplicative structure of the C^* -algebra.

Definition 4.5. A compact quantum metric space (\mathfrak{A}, L) is *Jordan-Lie* when L is lower semi-continuous, and its domain $\text{dom}(L)$ is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{A})$.

The Lipschitz distance between Jordan-Lie quantum compact metric spaces is actually achieved, as established in the following lemma. This observation will prove useful in establishing that the Lipschitz distance is indeed a distance up to quantum isometry.

Lemma 4.6. If $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two Jordan-Lie compact quantum metric spaces such that $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) < \infty$ then there exists a $*$ -isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that:

$$\max \left\{ |\ln(\text{dil}(\varphi))|, |\ln(\text{dil}(\varphi^{-1}))| \right\} = \text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})).$$

Proof. Suppose that $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = C$ for some $C \geq 0$. There exists a sequence of $*$ -isomorphism $(\varphi_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have:

$$C^{-1} \exp \left(-\frac{1}{n+1} \right) L_{\mathfrak{B}} \circ \varphi_n \leq L_{\mathfrak{A}} \leq C \exp \left(\frac{1}{n+1} \right) L_{\mathfrak{B}} \circ \varphi_n.$$

Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $L_{\mathfrak{A}}(a) < \infty$. Since $\|\varphi_n(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$ and $L_{\mathfrak{B}} \circ \varphi_n(a) \leq 2CL_{\mathfrak{A}}(a)$ for all $n \in \mathbb{N}$, we conclude that $(\varphi_n(a))_{n \in \mathbb{N}}$ admits a convergent subsequence since $L_{\mathfrak{B}}$ is a lower semicontinuous Lip-norm. Let $\varphi_{\infty}(a)$ be its limit.

Since $\{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq n, \|a\| \leq n\}$ is compact for the norm $\|\cdot\|_{\mathfrak{A}}$, hence separable for all $n \in \mathbb{N}$, so is:

$$\text{dom}(L_{\mathfrak{A}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) < \infty\} = \bigcup_{n \in \mathbb{N}} \{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq n, \|a\|_{\mathfrak{A}} \leq n\}.$$

Let \mathfrak{F} be a countable dense subset of $\{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) < \infty\}$. A diagonal argument proves that there exists a subsequence $(\varphi_{f(n)})_{n \in \mathbb{N}}$ such that for all $a \in \mathfrak{F}$ we have $(\varphi_{f(n)}(a))_{n \in \mathbb{N}}$ converges uniformly to $\varphi_{\infty}(a)$ (see [20, Theorem 5.13]).

Moreover, if $a \in \mathfrak{sa}(\mathfrak{A})$ with $L_{\mathfrak{A}}(a) < \infty$, then for all $\varepsilon > 0$, there exists $a_{\varepsilon} \in \mathfrak{F}$ with $\|a - a_{\varepsilon}\| < \frac{\varepsilon}{3}$. Let $N \in \mathbb{N}$ be such that for all $p, q \geq N$, we have $\|\varphi_{f(p)}(a_{\varepsilon}) - \varphi_{f(q)}(a_{\varepsilon})\|_{\mathfrak{B}} \leq \frac{\varepsilon}{3}$. Thus for all $p, q \geq N$, we have:

$$\begin{aligned} |\varphi_{f(p)}(a) - \varphi_{f(q)}(a)| &\leq |\varphi_{f(p)}(a - a_{\varepsilon})| + |\varphi_{f(p)}(a_{\varepsilon}) - \varphi_{f(q)}(a_{\varepsilon})| + |\varphi_{f(q)}(a - a_{\varepsilon})| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus $(\varphi_{f(n)}(a))_{n \in \mathbb{N}}$ converges as well, since it is a Cauchy sequence in \mathfrak{A} which is complete. Its limit is denoted once more by $\varphi_{\infty}(a)$.

Note that since for all $n \in \mathbb{N}$ and for all $a \in \text{dom}(L_{\mathfrak{A}})$, we have $\|\varphi_n(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$, we also have $\|\varphi_{\infty}(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$. We thus have defined an isometric map $\varphi_{\infty} : \text{dom}(L_{\mathfrak{A}}) \rightarrow \mathfrak{sa}(\mathfrak{B})$. Moreover, as a pointwise limit of Jordan-Lie morphisms, φ_{∞} is also a Jordan-Lie morphisms on $\text{dom}(L)$.

Now $L_{\mathfrak{B}}$ is lower semi-continuous and, for all $n \in \mathbb{N}$ we have $L_{\mathfrak{B}} \circ \varphi_n(a) \leq C \exp(\frac{1}{n+1}) L_{\mathfrak{A}}(a)$. Thus $L_{\mathfrak{B}} \circ \varphi_{\infty}(a) \leq C L_{\mathfrak{A}}(a)$. Thus $\text{dil}(\varphi_{\infty}) \leq C$.

Thus φ_{∞} extends by continuity to a Jordan-Lie morphism from $\mathfrak{sa}(\mathfrak{A})$ to $\mathfrak{sa}(\mathfrak{B})$. Our argument is now concluded in the same manner as [20, Claim 5.18, Theorem 5.13] and proves that φ_{∞} extends to a unital *-morphism from \mathfrak{A} to \mathfrak{B} with $\text{dil}(\varphi_{\infty}) \leq C$.

The same method may be applied to construct some subsequence of $(\varphi_n^{-1})_{n \in \mathbb{N}}$ converging pointwise on $\text{dom}(L_{\mathfrak{B}})$ to some *-morphism ψ_{∞} on \mathfrak{B} with $L_{\mathfrak{A}} \circ \psi_{\infty} \leq C L_{\mathfrak{B}}$. Up to extracting further subsequences, we shall henceforth assume that both $(\varphi_{f(n)})_{n \in \mathbb{N}}$ and $(\varphi_{f(n)}^{-1})_{n \in \mathbb{N}}$ converge pointwise to, respectively φ_{∞} on $\text{dom}(L_{\mathfrak{A}})$ and ψ_{∞} on $\text{dom}(L_{\mathfrak{B}})$. It is then immediate to check that $\varphi_{\infty} \circ \psi_{\infty}$ is the identity of $\text{dom}(L_{\mathfrak{B}})$ and $\psi_{\infty} \circ \varphi_{\infty}$ is the identity on $\text{dom}(L_{\mathfrak{A}})$. Then by construction, $\psi_{\infty} \circ \varphi_{\infty}$ is the identity on \mathfrak{A} and $\varphi_{\infty} \circ \psi_{\infty}$ is the identity on \mathfrak{B} . Thus φ_{∞} is a *-isomorphism from \mathfrak{A} to \mathfrak{B} .

In particular, we also obtain that $L_{\mathfrak{A}} \circ \varphi_{\infty}^{-1} \leq C L_{\mathfrak{B}}$ and thus $\text{dil}(\varphi_{\infty}^{-1}) \leq C$. As we may not have both $\text{dil}(\varphi) < C$ and $\text{dil}(\varphi^{-1}) < C$, since C is the infimum of the dilations of such *-isomorphisms, the lemma is proven. \square

We now establish that the Lipschitz distance is indeed, a distance up to quantum isometry, and that it dominates the quantum propinquity.

Theorem 4.7. *The Lipschitz distance is an extended metric up to quantum isometry on the class of Jordan-Lie quantum compact metric spaces. Explicitly, for all $(\mathfrak{A}, L_{\mathfrak{A}})$, $(\mathfrak{B}, L_{\mathfrak{B}})$ and $(\mathfrak{D}, L_{\mathfrak{D}})$ Jordan-Lie compact quantum metric spaces, we have:*

- (1) $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \in [0, \infty]$, and is finite if and only if there exists a *-isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\varphi(\text{dom}(L_{\mathfrak{A}})) = \text{dom}(L_{\mathfrak{B}})$,

- (2) $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{D}, L_{\mathfrak{D}})) \leq \text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) + \text{LipD}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{D}, L_{\mathfrak{D}})),$
- (3) $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \text{LipD}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{A}, L_{\mathfrak{A}})),$
- (4) $\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = 0$ if and only if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are isometrically isomorphic, i.e. there exists a *-isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $L_{\mathfrak{B}} \circ \varphi = L_{\mathfrak{A}}$ on $\text{sa}(\mathfrak{A})$. Such a map φ is called a quantum isometry or an isometric isomorphism.
- (5) if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are two F -quasi-Leibniz quantum compact metric spaces then:

$$\Lambda_F((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq |1 - \exp(\text{LipD}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})))| \left(\frac{1}{2} + \max \{ \text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}}), \text{diam}(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}}) \} \right).$$

Proof. The function LipD is valued in $[0, \infty]$ by definition, and finite if and only if there exists a bi-Lipschitz isomorphism between its two arguments, by Corollary (2.3). It is symmetric in its two arguments by construction. The triangle inequality follows from simple computations as well.

The last assertion of our proposition follows immediately from [18, Proposition 3.80].

If there exists an isometric isomorphism between two compact quantum metric spaces, then their Lipschitz distance is null. Only the converse of this observation requires our assumption that the domain of Lip-norms be Jordan-Lie algebras. We simply apply Lemma (4.6). \square

We now prove that closed balls for the Lipschitz distance are compact classes for the dual propinquity.

Theorem 4.8. *Let $(\mathfrak{A}, L_{\mathfrak{A}})$ be a F -quasi-Leibniz quantum compact metric space for some admissible function F . If $R \geq 0$ then the class \mathcal{B} of F -quasi-Leibniz quantum compact metric spaces in the closed ball of center $(\mathfrak{A}, L_{\mathfrak{A}})$ and radius R for the Lipschitz distance LipD is totally bounded for the quantum Gromov-Hausdorff propinquity.*

Therefore, the closure of \mathcal{B} for the dual propinquity is compact.

Proof. For all (\mathfrak{B}, L) within Lipschitz distance R of $(\mathfrak{A}, L_{\mathfrak{A}})$, there exists by definition a *-isomorphism $\varphi_{\mathfrak{B}} : \mathfrak{A} \rightarrow \mathfrak{B}$ which maps the domain of $L_{\mathfrak{A}}$ into the domain of L . We set $L' = L \circ \varphi_{\mathfrak{B}}$ and note that L' is a lower semicontinuous Lip-norm with the same domain as $L_{\mathfrak{A}}$. Moreover, by definition of the Lipschitz distance, we have $L_{\mathfrak{A}} \leq \exp(R)L'$. Last, (\mathfrak{A}, L') and (\mathfrak{B}, L) are isometrically isomorphic, thus their propinquity is zero, and so is their Lipschitz distance.

Thus by Proposition (3.6), the closed ball \mathcal{B} of center $(\mathfrak{A}, L_{\mathfrak{A}})$ and radius R is totally bounded for $\text{Haus}_{\mathfrak{A}}^{\circ}$. By Proposition (3.5), the subclass of F -quasi-Leibniz quantum compact metric spaces in the closed ball \mathcal{B} is totally bounded for the quantum propinquity. The rest of the theorem follows from the completeness of the dual propinquity [17] and the dominance of the quantum propinquity over the dual propinquity. \square

We conclude this section by proving that the Lipschitz distance is indeed complete.

Theorem 4.9. *The distance LipD is complete on the class of Jordan-Lie quantum compact metric spaces.*

Proof. As in the proof of Theorem (4.8), we can assume that we are given a sequence $(L_n)_{n \in \mathbb{N}}$ of lower semi-continuous Lip-norms on some unital C^* -algebra, such that $(\mathfrak{A}, L_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for LipD.

By Theorem (4.7), the sequence $\mathfrak{L}_n = \{a \in \mathfrak{sa}(\mathfrak{A}) : L_n(a) \leq 1\}$ is Cauchy for $\text{Haus}_{\|\cdot\|_{\mathfrak{A}}}^\circ$. As the latter metric is complete, $(\mathfrak{L}_n)_{n \in \mathbb{N}}$ converges to some \mathfrak{L} .

Let $\varepsilon \in (0, 1)$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have:

$$(1 - \varepsilon)L_q \leq L_p \leq (1 + \varepsilon)L_q$$

for all $p, q \geq N$. In other words, $\mathfrak{L}_q \subseteq (1 + \varepsilon)\mathfrak{L}_p$ and $\mathfrak{L}_p \subseteq \frac{1}{1 - \varepsilon}\mathfrak{L}_q$ for all $p, q \geq N$. Now $(1 + \varepsilon)\mathfrak{L}_n$ is closed for all $n \in \mathbb{N}$, thus the hyperspace of its closed subsets is complete and thus closed for the Hausdorff distance $\text{Haus}_{\|\cdot\|_{\mathfrak{A}}}$. Consequently, $\mathfrak{L} \subseteq (1 + \varepsilon)\mathfrak{L}_n$ for all $n \geq N$. Thus $L_n \leq (1 + \varepsilon)L$.

On the other hand, let $a \in \mathfrak{L}_n$ for $n \geq N$. For $p \geq N$ we have $a \in \frac{1}{1 - \varepsilon}\mathfrak{L}_p$ so $a \in \frac{1}{1 - \varepsilon}\mathfrak{L}$ since $\frac{1}{1 - \varepsilon}\mathfrak{L}$ is the Hausdorff limit of $s = \left(\frac{1}{1 - \varepsilon}\mathfrak{L}_p\right)_{p \geq n}$ and thus contains all the limits of convergent sequences obtained by picking one element in each set of s .

We conclude $\text{LipD}((\mathfrak{A}, L), (\mathfrak{A}, L_n)) \leq \ln(1 + \varepsilon)$ for $n \geq N$. Our theorem is now proven. \square

5. A METRIC FOR POINTWISE CONVERGENCE ON THE AUTOMORPHISM GROUP OF A QUANTUM COMPACT METRIC SPACE

We now introduce a new metric on the automorphism group of a quantum compact metric space. Our motivation is given by Theorem (2.1), and in particular Corollary (2.4), as well as our new understanding of compactness for Lip-norms with a shared domain.

Definition 5.1. Let (\mathfrak{A}, L) be a quantum compact metric space. For any $*$ -automorphism of \mathfrak{A} , we define:

$$\text{mk}\ell_L(\alpha) = \sup \{\|\alpha(a) - a\|_{\mathfrak{A}} : a \in \text{dom}(L), L(a) \leq 1\}.$$

Proposition 5.2. *Let (\mathfrak{A}, L) be a quantum compact metric space.*

- (1) *$\text{mk}\ell_L$ is a length function on $\text{Aut}(\mathfrak{A})$ which metrizes the topology of pointwise convergence,*
- (2) *$\text{diam}(\text{Aut}(\mathfrak{A}), \text{mk}\ell_L) \leq \text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_L)$*
- (3) *If L is closed, Ξ is a subset of $\text{Aut}(\mathfrak{A})$ and there exists $C > 0$ such that:*

$$\forall \alpha \in \Xi \quad L \leq CL \circ \alpha$$

then Ξ is totally bounded for $\text{mk}\ell_L$.

Proof.

Claim 5.3. *$\text{mk}\ell_L$ is a length function on the group $\text{Aut}(\mathfrak{A})$.*

if $\text{id}_{\mathfrak{A}}$ is the identity of \mathfrak{A} then $\text{mk}\ell_L(\text{id}_{\mathfrak{A}}) = 0$. Now let $\alpha \in \text{Aut}(\mathfrak{A})$.

If $\text{mk}\ell_L(\alpha) = 0$ then $\|a - \alpha(a)\|_{\mathfrak{A}} = 0$ for all $a \in \text{dom}(L)$ with $L(a) \leq 1$. Thus $\|a - \alpha(a)\|_{\mathfrak{A}} = 0$ for all $a \in \text{dom}(L)$, and by continuity of α and density of $\text{dom}(L)$,

we conclude that $\|a - \alpha(a)\|_{\mathfrak{A}} = 0$ for all $a \in \mathfrak{sa}(\mathfrak{A})$. Thus by linearity, $\alpha(a) = a$ for all $a \in \mathfrak{A}$.

Moreover, since α is an isometry of $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$, we have for all $a \in \mathfrak{A}$:

$$\|a - \alpha(a)\|_{\mathfrak{A}} = \|\alpha(\alpha^{-1}(a) - a)\|_{\mathfrak{A}} = \|\alpha^{-1}(a) - a\|_{\mathfrak{A}}$$

from which it follows that $\text{mk}\ell_{\mathbb{L}}(\alpha) = \text{mk}_{\mathbb{L}}(\alpha^{-1})$.

Let now $\beta \in \text{Aut}(\mathfrak{A})$. For all $a \in \mathfrak{A}$:

$$\begin{aligned} \|\alpha \circ \beta(a) - a\|_{\mathfrak{A}} &\leq \|\alpha \circ \beta(a) - \alpha(a)\|_{\mathfrak{A}} + \|\alpha(a) - a\|_{\mathfrak{A}} \\ &\leq \|\alpha(\beta(a) - a)\|_{\mathfrak{A}} + \|\alpha(a) - a\|_{\mathfrak{A}} \\ &\leq \|\beta(a) - a\|_{\mathfrak{A}} + \|\alpha(a) - a\|_{\mathfrak{A}}, \end{aligned}$$

from which we conclude:

$$\text{mk}\ell_{\mathbb{L}}(\alpha \circ \beta) \leq \text{mk}\ell_{\mathbb{L}}(\beta) + \text{mk}\ell_{\mathbb{L}}(\alpha).$$

Claim 5.4. *The topology induced by $\text{mk}\ell_{\mathbb{L}}$ is the topology of pointwise convergence.*

Let $(\alpha_j)_{j \in J}$ be a net in $\text{Aut}(\mathfrak{A})$ which converges pointwise to some $\alpha \in \text{Aut}(\mathfrak{A})$ over \mathfrak{A} , where (J, \succ) is a directed set. Let $\varepsilon > 0$.

Let $\varphi \in \mathcal{S}(\mathfrak{A})$. Since $\mathfrak{B} = \{a \in \text{dom}(\mathbb{L}) : \mathbb{L}(a) \leq 1, \varphi(a) = 0\}$ is totally bounded, there exists $\mathfrak{F} \subseteq \mathfrak{B}$ with \mathfrak{F} finite and such that for all $a \in \mathfrak{B}$ there exists $f(a) \in \mathfrak{F}$ with $\|a - f(a)\|_{\mathfrak{A}} \leq \frac{\varepsilon}{3}$.

For each $a \in \mathfrak{F}$, there exists $j_a \in J$ such that if $j \succ j_a$, we have $\|\alpha_j(a) - \alpha(a)\|_{\mathfrak{A}} \leq \frac{\varepsilon}{3}$. As J is directed and \mathfrak{F} is finite, there exists $j' \in J$ with $j' \succ j_a$ for all $a \in \mathfrak{F}$. Thus, if $a \in \mathfrak{B}$ and $j \succ j'$, then:

$$\begin{aligned} \|\alpha_j(a) - \alpha(a)\|_{\mathfrak{A}} &\leq \|\alpha_j(a) - \alpha_j(f(a))\|_{\mathfrak{A}} + \|\alpha_j(f(a)) - \alpha(f(a))\|_{\mathfrak{A}} \\ &\quad + \|\alpha(f(a)) - \alpha(a)\|_{\mathfrak{A}} \leq \varepsilon, \end{aligned}$$

where we used that all automorphisms are isometries. Thus, for all $j \succ j'$ we have:

$$\text{mk}\ell_{\mathbb{L}}(\alpha_j^{-1} \circ \alpha) \leq \varepsilon.$$

Conversely, let $(\alpha_j)_{j \in J}$ be a net in $\text{Aut}(\mathfrak{A})$ converging for $\text{mk}\ell_{\mathbb{L}}$ to $\alpha \in \text{Aut}(\mathfrak{A})$. Let $a \in \mathfrak{sa}(\mathfrak{A})$ and $\varepsilon > 0$.

Since $\text{dom}(\mathbb{L})$ is dense, there exists $b \in \text{dom}(\mathbb{L})$ such that $\|a - b\|_{\mathfrak{A}} \leq \frac{\varepsilon}{3}$. Now, by assumption, there exists $j_0 \in J$ such that for all $j \succ j_0$, we have $\text{mk}\ell_{\mathbb{L}}(\alpha_j^{-1} \circ \alpha) \leq \frac{\varepsilon}{3(\mathbb{L}(b)+1)}$. Thus $\|\alpha_j(b) - \alpha(b)\|_{\mathfrak{A}} \leq \frac{\varepsilon}{3}$ and thus for all $j \succ j_0$:

$$\|\alpha_j(a) - \alpha(a)\|_{\mathfrak{A}} \leq \|\alpha_j(a) - \alpha_j(b)\|_{\mathfrak{A}} + \|\alpha_j(b) - \alpha(b)\|_{\mathfrak{A}} + \|\alpha(b) - \alpha(a)\|_{\mathfrak{A}} \leq \varepsilon,$$

as desired.

Claim 5.5. *Assume \mathbb{L} is closed. The set $\{\alpha \in \text{Aut}(\mathfrak{A}) : \mathbb{L} \circ \alpha \leq C\mathbb{L}\}$, for some fixed $C > 0$, is totally bounded and closed for $\text{mk}\ell_{\mathbb{L}}$.*

Fix $\varphi \in \mathcal{S}(\mathfrak{A})$.

All automorphisms of \mathfrak{A} are isometries and thus $\text{Aut}(\mathfrak{A})$ is equicontinuous over the compact set $\mathfrak{X} = \{a \in \text{dom}(\mathbb{L}) : \mathbb{L}(a) \leq 1, \varphi(a) = 0\}$. Moreover, for all $a \in \text{dom}(\mathbb{L})$ with $\mathbb{L}(a) \leq 1$, then $\mathbb{L} \circ \alpha(a) \leq C$. Thus, $\alpha(a) \in \{a \in \text{dom}(\mathbb{L}) : \mathbb{L}(a) \leq C, \varphi(a) = 0\}$ for all $a \in \text{dom}(\mathbb{L})$ with $\mathbb{L}(a) \leq 1$ with $\varphi \circ \alpha^{-1}(a) = 0$.

Thus, by Arzela-Ascoli, the set $\{\alpha \in \text{Aut}(\mathfrak{A}) : L \circ \alpha \leq CL\}$ is totally bounded for the supremum norm over \mathfrak{X} .

We note that as automorphisms are unital, the supremum over \mathfrak{X} of the difference of two automorphisms of \mathfrak{A} equals to the supremum over $\{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) \leq 1\}$. Therefore, $\{\alpha \in \text{Aut}(\mathfrak{A}) : L \circ \alpha \leq CL\}$ is totally bounded for $\text{mk}\ell_L$.

Moreover, if $L \circ \alpha_n \leq CL$ for all $n \in \mathbb{N}$, by lower semicontinuity, $L \circ \alpha \leq CL$ as well. Thus our set is closed. As $\text{mk}\ell_L$ is complete, our proof is complete. \square

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